Braid structure and raising-lowering operator formalism in Sutherland model

1998 J. Phys. A: Math. Gen. 31 L473

(http://iopscience.iop.org/0305-4470/31/25/002)

J. Phys. A: Math. Gen. **31** (1998) L473–L480. Printed in the UK PII: S0305-4470(98)91503-7

LETTER TO THE EDITOR

Braid structure and raising-lowering operator formalism in Sutherland model

Akira Takamura† and Ken'ichi Takano‡

† Department of Physics, Nagoya University, Nagoya 464-8602, Japan
‡ Laboratory of Theoretical Condensed Matter Physics and Research Center for Advanced Photon Technology, Toyota Technological Institute, Nagoya 468-8511, Japan

Received 28 January 1998, in final form 30 April 1998

Abstract. We algebraically construct the Fock space of the Sutherland model in terms of the eigenstates of the pseudomomenta as basis vectors. For this purpose, we derive the raising and lowering operators which increase and decrease eigenvalues of pseudomomenta. The operators exchanging eigenvalues of two pseudomomenta have been known. All the eigenstates are systematically produced by starting from the ground state and multiplying these operators by it.

The Sutherland model is a solvable quantum many-body system with inverse-square interaction on a circumference [1]. The ground-state wavefunction is of the Jastrow type and excited states are polynomials multiplied by the ground state. Among the polynomials, the symmetric ones are Jack polynomials [2–4], while the others are called nonsymmetric Jack polynomials. These energy eigenstates can be taken as eigenstates of the pseudomomenta [5, 6], which commute with each other and with the Hamiltonian.

For its rich content, the Sutherland model has been zealously investigated at various standpoints. For example, the Sutherland model is regarded as a model which describes the edge state in the fractional quantum Hall effect [7]. It may describe the fractional statistics of quasiparticles [8]. Also a deep connection of this model to the conformal field theory is found [9]. Haldane argued that the Sutherland model is equivalent to the system of particles obeying the exclusion statistics if the coupling constant is a rational number [10]. Based on this assumption he obtained the concrete form of the two-point correlation function; i.e. as intermediate states, he only used free particle states obeying the exclusion statistics. The result coincides with the exact one which was calculated by using the duality of Jack polynomials [11–14]. The duality means the invariance of the Jack polynomials under a nonlinear transformation with the replacement of the coupling constant by its inverse. In the Sutherland model, many interesting properties such as the exclusion statistics are deduced by directly inspecting the Jack polynomials.

To deeply understand the Sutherland model, we need to reformulate algebraically the eigenvalue problem of this model. We mention its importance by recalling the case of a harmonic oscillator. Although this problem is solved in terms of Hermite polynomials, the algebraic approach using creation and annihilation operators revealed the essence of the model. The quantum field theory is formulated on the basis of harmonic oscillators. In the Calogero model, with inverse-square interaction and harmonic potential, creation and annihilation operators are examined [15, 16]. In the Sutherland model, a hopeful algebraic

0305-4470/98/250473+08$19.50

approach means that a simple and transparent algebra determines all the energy levels and their degeneracy. There are some algebraic treatments for symmetric [17] and nonsymmetric Jack polinomials [18, 19], where a polynomial generates another one by some operations. However such a generated state is not an eigenstate of the pseudomomenta except for special cases and is not simple for the present purpose to seek a physical transparency.

In this letter, we propose a novel algebraic formalism for the eigenvalue problem in the Sutherland model. The formalism is based on operators which increase, decrease and exchange the eigenvalues of psudemomenta. The raising and lowering operators are derived in this letter and the operator for exchange has been introduced [19]. Starting from the ground state, we can reach an arbitrary eigenstate of the pseudomomenta by multiplying a finite number of operators. The Fock space of the Sutherland model is reproduced in terms of eigenstates of the pseudomomenta.

We consider N particles on a circumference with length π and denote the coordinate of the ith particle by θ_i. For these particles we introduce an operator K_{ij} $(i \neq j)$ which exchanges coordinates θ_i and θ_j, i.e. $K_{ij}\theta_i = \theta_j K_{ij}$. Then an extended version of the Sutherland model is given by the Hamiltonian

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial\theta_i^2} + \frac{1}{2}\sum_{i\neq j} \frac{\beta(\beta - K_{ij})}{\sin^2[(\theta_i - \theta_j)/2]} \tag{1}$$

where β is the coupling constant. This Hamiltonian is invariant against the exchange of the coordinates of particles and satisfies the commutation relation $[H, K_{ij}] = 0$. To make the description simple, we use the complex coordinate $z_i = \exp(\mathrm{i}\theta_i)$ instead of θ_i. The momentum is accordingly represented as

$$p_i = z_i \frac{\partial}{\partial z_i}. \tag{2}$$

The quantization condition is then given by

$$[p_i, z_j] = \delta_{ij} z_i. \tag{3}$$

The Hamiltonian (1) is rewritten as

$$H = \sum_{i=1}^{N} p_i^2 + \sum_{i,j} \frac{z_i z_j}{(z_i - z_j)^2} \beta(\beta - K_{ij}). \tag{4}$$

Dunkl [5] and Cherednik [6] introduced the pseudomomentum which is defined as

$$D_i = p_i + \beta \sum_{j(>i)} \frac{z_i}{z_i - z_j} K_{ij} - \beta \sum_{j(<i)} K_{ij} \frac{z_i}{z_i - z_j}. \tag{5}$$

In terms of $\{D_i\}$, the Hamiltonian and the total momentum are written as

$$H = \sum_{i=1}^{N} D_i^2 \qquad P = \sum_{i=1}^{N} D_i. \tag{6}$$

The pseudomomenta are Hermitian $(D_i^\dagger = D_i)$ and commute with each other:

$$[D_i, D_j] = 0. \tag{7}$$

Hence they also commute with the Hamiltonian ($[H, D_i] = 0$). The exchange operators affect the pseudomomenta through the relations

$$D_i K_{i,i+1} - K_{i,i+1} D_{i+1} = \beta, \tag{8}$$

$$[D_j, K_{i,i+1}] = 0 \qquad (j \neq i, i+1). \tag{9}$$

The quantization condition (3) is represented as

$$[D_i, z_j] = \begin{cases} z_i + \beta z_i \sum_{j(<i)} K_{ij} + \beta \sum_{j(>i)} K_{ij} z_i & (i = j) \\ -\beta\{z_j K_{ij}\theta(i-j) + K_{ij} z_j \theta(j-i)\} & (i \neq j). \end{cases} \tag{10}$$

Here the step function $\theta(x)$ is 1 for $x \geqslant 0$ and 0 otherwise. While the Hamiltonian (6) is of the form for free particles with momenta $\{D_i\}$, the quantization condition (10) is rather complicated. That is, all the effects of the long-range interaction are involved in the quantization condition (10). For this reason the interaction in the Hamiltonian (1) is called a statistical interaction. The operators $\{D_i, z_j, K_{kl}\}$ are closed with respect to their mutual products, and thereby forming an algebra. However it is not a Lie algebra, since the commutator of some operators is no longer represented by a linear combination of the operators. Relations (7)–(10) form a degenerate double-affine Hecke algebra. The same structure for the Calogero model was examined by Ujino and Wadati [15] and Kakei [16].

We construct the energy eigenvalues and the eigenstates of the Sutherland model in a completely algebraic manner. First, we examine the operator $X_{i,i+1}$ defined by

$$X_{i,i+1} = \mathrm{i}[D_i, K_{i,i+1}] \qquad (i = 1, \ldots, N-1) \tag{11}$$

which is clearly Hermitian ($X^{\dagger}_{i,i+1} = X_{i,i+1}$). We call this the *braid-exclusion operator*. The q-deformed version of this operator was first introduced by Killirov and Noumi [19]. Relations (8) and (9) for D_i and $K_{i,i+1}$ are converted to the following relations:

$$D_i X_{i,i+1} = X_{i,i+1} D_{i+1} \tag{12}$$

$$D_{i+1} X_{i,i+1} = X_{i,i+1} D_i \tag{13}$$

$$[D_k, X_{i,i+1}] = 0 \qquad (k \neq i, i+1). \tag{14}$$

These equations mean that $X_{i,i+1}$ exchanges the pseudomomenta D_i and D_{i+1}.

From definition (11) the square of $X_{i,i+1}$ is written as

$$X^2_{i,i+1} = (D_i - D_{i+1})^2 - \beta^2. \tag{15}$$

The positive semidefiniteness of $X^2_{i,i+1}$ requires that the difference of eigenvalues of D_i and D_{i+1} must differ by a number larger than or equal to $|\beta|$. As will be clear by later examination, any eigenvalues of the pseudomomenta are integers in both the special cases of $|\beta| = 0$ and 1. For $|\beta| = 0$, the particles are bosonic since (15) shows that their eigenvalues can take the same value. On the other hand, for $|\beta| = 1$, the particles are fermionic since the eigenvalues must take different integers due to (15). Thus, relation (15) for $0 < |\beta| < 1$ shows neither bosonic nor fermionic statistics but suggests Haldane's exclusion statistics [20, 21].

The braid-exclusion operators satisfy the following relations:

$$X_{i,i+1} X_{i+1,i+2} X_{i,i+1} = X_{i+1,i+2} X_{i,i+1} X_{i+1,i+2} \tag{16}$$

$$X_{i,i+1} X_{j,j+1} = X_{j,j+1} X_{i,i+1} \qquad (|i-j| \geqslant 2) \tag{17}$$

which are derived from definition (11) and relations (7)–(9) by straightforward calculation. Equations (16) and (17) are the very relations which generators of a braid group satisfy [22]; equation (16) is also of the same form as the Yang–Baxter relation. They essentially determine the characters of operators which will be introduced below. Thus the exchange operator $X_{i,i+1}$ for the pseudomomenta possesses both the characters of the exclusion statistics and the braid group structure. This is the reason why we have called them braid-exclusion operators. The operator, however, has no inverse operator against any true generators for a braid group. In fact the exclusion character (15) allows the eigenvalue of $X_{i,i+1}$ to vanish when the eigenvalue of D_i differs from that of D_{i+1} by $\pm\beta$.

Next we recall an operator $e^\dagger$ which is defined as

$$e^\dagger = K_{N,N-1}K_{N-1,N-2}\ldots K_{32}K_{21}z_1 \tag{18}$$

and call it the *displacement operator*. It was introduced by Knop and Sahi [18] in relation to nonsymmetric Jack polynomials. The equation $|z_i| = 1$ guarantees its unitarity:

$$e^\dagger e = ee^\dagger = 1. \tag{19}$$

Equations (7)–(10) show that the operator $e^\dagger$ satisfy the relations

$$D_j e^\dagger - e^\dagger D_{j+1} = 0 \qquad (j = 1, \ldots, N-1) \tag{20}$$

$$D_N e^\dagger - e^\dagger D_1 = e^\dagger. \tag{21}$$

That is, $e^\dagger$ displaces all the subscripts of D_i by one periodically. Equations (7)–(10) also deduce the relation among $e^\dagger$ and $\{X_{i,i+1}\}$:

$$X_{i,i+1}e^\dagger = e^\dagger X_{i+1,i+2} \qquad (i = 1, \ldots, N-2) \tag{22}$$

$$X_{N-1,N}(e^\dagger)^2 = (e^\dagger)^2 X_{12}. \tag{23}$$

These equations show that $e^\dagger$ also displaces all the subscripts of the braid-exclusion operators by 1.

Before constructing raising and lowering operators, we introduce an operator

$$a_i^\dagger = X_{i,i+1}X_{i+1,i+2}\ldots X_{N-1,N}e^\dagger \qquad (i = 1, \ldots, N) \tag{24}$$

as an intermediate. In the case of $i = N$ this equation reads as $a_N^\dagger = e^\dagger$. We call $a_i^\dagger$ the *constituent operator*. The constituent operators and the pseudomomenta satisfy the relations:

$$D_j a_i^\dagger - a_i^\dagger D_{j+1} = 0 \qquad (1 \leqslant j \leqslant i-1) \tag{25}$$

$$D_i a_i^\dagger - a_i^\dagger D_1 = a_i^\dagger \tag{26}$$

$$[D_j, a_i^\dagger] = 0 \qquad (i+1 \leqslant j \leqslant N) \tag{27}$$

which are derived from (12)–(14), (20) and (21). The constituent operators and the braid-exclusion operators satisfy the relations:

$$X_{i,i+1}a_{i+1}^\dagger = a_i^\dagger \tag{28}$$

$$X_{i,i+1}a_j^\dagger = \begin{cases} a_j^\dagger X_{i+1,i+2} & (j \geqslant i+2) \\ a_j^\dagger X_{i,i+1} & (j \leqslant i-1) \end{cases} \tag{29}$$

$$a_i^\dagger a_j^\dagger = a_j^\dagger a_{i+1}^\dagger X_{12} \qquad (j \geqslant i+1) \tag{30}$$

which are derived from (16), (17), (22) and (23). Number-like operators $a_i^\dagger a_i$ and $a_i a_i^\dagger$ are expressed in terms of the pseudomomenta as follows

$$a_i^\dagger a_i = \prod_{m=i+1}^{N} [(D_i - D_m)^2 - \beta^2] \qquad (1 \leqslant i \leqslant N-1) \tag{31}$$

$$a_i a_i^\dagger = \prod_{m=i+1}^{N} [(D_1 - D_m + 1)^2 - \beta^2] \qquad (1 \leqslant i \leqslant N-1) \tag{32}$$

$$a_N^\dagger a_N = a_N a_N^\dagger = 1 \tag{33}$$

which are derived from (12)–(15) and (19)–(21).

The *raising operator* is defined as a simple power of a constituent operator:

$$b_i^\dagger = (a_i^\dagger)^i \qquad (i = 1, \dots, N) \tag{34}$$

and the corresponding *lowering operator* is its Hermitian conjugate. The raising operators and the pseudomomenta satisfies the commutation relations:

$$[D_i, b_j^\dagger] = \theta(j - i) b_j^\dagger \tag{35}$$

as is derived from (25)–(27). That is, $b_j^\dagger$ raises by 1 the eigenvalues of pseudomomenta with subscript i for $i \leqslant j$ and is qualified to be called a raising operator. The raising operators are boson-like since they commute with each other:

$$[b_i^\dagger, b_j^\dagger] = 0 \tag{36}$$

which are derived from (28)–(30).

Number-like operators are expressed in terms of the pseudomomenta as:

$$b_i^\dagger b_i = \prod_{l=1}^{i} \prod_{m=i+1}^{N} [(D_l - D_m)^2 - \beta^2] \qquad (1 \leqslant i \leqslant N-1) \tag{37}$$

$$b_i b_i^\dagger = \prod_{l=1}^{i} \prod_{m=i+1}^{N} [(D_l - D_m + 1)^2 - \beta^2] \qquad (1 \leqslant i \leqslant N-1) \tag{38}$$

$$b_N^\dagger b_N = b_N b_N^\dagger = 1 \tag{39}$$

which are derived from (25)–(27) and (31)–(33). Further, (28)–(30) yields the following relations:

$$X_{i,i+1} b_j^\dagger = b_j^\dagger X_{i,i+1} \qquad (i \neq j) \tag{40}$$

$$b_i^\dagger X_{i,i+1} b_i^\dagger = [(D_{i+1} - D_i + 1)^2 - \beta^2] X_{i,i+1} b_{i-1}^\dagger b_{i+1}^\dagger. \tag{41}$$

We now construct the Fock space of the Sutherland model by using the set of operators $\{D_i, b_j, X_{kl}\}$. Concretely, we produce all the eigenstates of $\{D_i\}$, starting from a state and multiplying operators $\{b_j, X_{kl}\}$ to it. These states are also eigenstates of the Hamiltonian because of the commutability of H and $\{D_i\}$. An eigenstate with different energy levels is produced by multiplying the raising or lowering operators, and a degenerate state is produced by multiplying the braid-exclusion operators.

We label an eigenstate of $\{D_i\}$ by their eigenvalues $\{k_i\}$ as

$$D_i |k_1, k_2, \dots, k_N\rangle = k_i |k_1, k_2, \dots, k_N\rangle \qquad (i = 1, \dots, N). \tag{42}$$

We start the construction with a state which has the eigenvalues $k_i = \alpha_i$ $(i = 1, \dots, N-1)$ and is annihilated by lowering operators as

$$b_i |\alpha_1, \alpha_2, \dots, \alpha_N\rangle = 0 \qquad (i = 1, \dots, N-1). \tag{43}$$

The case of $i = N$ is excluded in this equation, since $b_N (= \mathrm{e}^N)$ is exceptionally unitary and does not annihilate any state. Equation (43) reduces to

$$X_{i,i+1} |\alpha_1, \alpha_2, \dots, \alpha_N\rangle = 0 \qquad (i = 1, \dots, N-1) \tag{44}$$

due to the definitions of a_i and b_i. We begin the construction of the Fock space with a state satisfying condition $\alpha_1 > \alpha_2 > \cdots > \alpha_N$. Then (44) reduces to

$$K_{i,i+1} |\alpha_1, \dots, \alpha_N\rangle = \mathrm{sgn}(\beta) |\alpha_1, \dots, \alpha_N\rangle \tag{45}$$

by using definition (11) of $X_{i,i+1}$ and the algebra, (7)–(9). Hence the state $|\alpha_1, \dots, \alpha_N\rangle$ is a symmetric (antisymmetric) function for $\beta > 0$ $(\beta < 0)$.

To examine possible values of $\{\alpha_i\}$, we operate another $X_{i,i+1}$ to (44). Then we see that $\{\alpha_i\}$ are related to each other since (44) and (15) yield the relation $(\alpha_i - \alpha_{i+1})^2 = \beta^2$. In reality there stands a stronger condition:

$$\alpha_i - \alpha_{i+1} = |\beta| \qquad (i = 1, \ldots, N-1) \tag{46}$$

which is obtained by a calculation with (7)–(9). This condition is rewritten as

$$\alpha_i = \alpha_0 + \frac{N+1-2\mathrm{i}}{2}|\beta| \qquad (i = 1, \ldots, N) \tag{47}$$

with undetermined constant α_0 $(-\frac{1}{2} < \alpha_0 < \frac{1}{2})$. This kind of undetermined constant always appears in quantum mechanics on S^1 [23]. Hereafter we choose it as $\alpha_0 = 0$ so that the total momentum P of this state vanishes. We write the state with $\alpha_0 = 0$ in (43) simply as $|0\rangle$:

$$|0\rangle \equiv \left|\frac{N-1}{2}|\beta|, \frac{N-3}{2}|\beta|, \ldots, -\frac{N-1}{2}|\beta|\right\rangle. \tag{48}$$

We will see that this state is the true ground state in the Fock space which we are going to construct.

We have a series of excited states when we operate raising operators to the ground state $|0\rangle$. By introducing a new notation, we write them as follows

$$|n_1, n_2, \ldots, n_N\rangle\rangle \equiv (b_1^\dagger)^{n_1-n_2}(b_2^\dagger)^{n_2-n_3}\ldots(b_N^\dagger)^{n_N}|0\rangle. \tag{49}$$

Here we must impose the constraint $n_i \geqslant n_{i+1}$ $(i = 1, \ldots, N-1)$ so that the power of $b_i^\dagger$ is positive; $b_i^\dagger$ $(i \neq N)$ generally has no inverse operator since $b_i^\dagger b_i$ has eigenvalue 0 as seen in (33). In contrast the power n_N of the last operator $b_N^\dagger$ is unrestricted because of its unitarity (39). The negative power of $b_N^\dagger$ is read as the positive power of b_N, i.e. $(b_N^\dagger)^n = (b_N)^{-n}$. The constraint is concisely written as

$$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N. \tag{50}$$

The states defined by (49) are eigenstates of the pseudomomenta as is shown by (35):

$$D_i|n_1, n_2, \ldots, n_N\rangle\rangle = \left(n_i + \frac{N+1-2\mathrm{i}}{2}|\beta|\right)|n_1, n_2, \ldots, n_N\rangle\rangle. \tag{51}$$

Hence $|n_1, n_2, \ldots, n_N\rangle\rangle$ is identified as

$$|n_1, n_2, \ldots, n_N\rangle\rangle = |k_1, k_2, \ldots, k_N\rangle \tag{52}$$

with eigenvalue $k_i = n_i + (N+1-2\mathrm{i})|\beta|/2$ for D_i $(i = 1, \ldots, N)$. The norm of this state is calculated as

$$\langle\langle n_1, \ldots, n_N|n_1, \ldots, n_N\rangle\rangle = \prod_{i=1}^{N-1}\prod_{l=1}^{i}\prod_{m=i+1}^{N}\prod_{r=1}^{n_i-n_{i+1}}[((m-l)\beta + r + n_{i+1} - n_m)^2 - \beta^2] \tag{53}$$

by means of relations (35)–(39).

Next we operate a braid-exclusion operator $X_{i,i+1}$ to the eigenstate (42) of $\{D_i\}$. Then relations (12)–(15) yields the following equation:

$$X_{i,i+1}|\ldots, k_i, k_{i+1}, \ldots\rangle = \sqrt{(k_{i+1}-k_i)^2 - \beta^2}|\ldots, k_{i+1}, k_i, \ldots\rangle. \tag{54}$$

Hence, if $|k_{i+1} - k_i| \neq |\beta|$, $X_{i,i+1}$ produces a new state in which eigenvalues k_i and k_{i+1} are exchanged. Equation (54) for states corresponds to relation (12) and (13) for operators, which means the exchange of D_i and D_{i+1}. Operating $\{X_{i,i+1}\}$ to $|k_1, k_2, \ldots, k_N\rangle$ in (52) finite times, we reach any possible order of $\{k_i\}$. Redefining k_i as the eigenvalue of D_i, possible eigenvalues of $\{D_i\}$ are written as

$$k_i = n_{\sigma(i)} + \frac{N+1-2\sigma(i)}{2}|\beta| \qquad (i = 1, \ldots, N) \tag{55}$$

where σ is a permutation among $1, 2, \ldots, N$ which satisfies $n_{\sigma(i)} \neq n_{\sigma(j)}$ for $|\sigma(i) - \sigma(j)| = 1$. For $|\beta| = 1$, constraint (50) is equivalent to the Pauli principle: $|k_i - k_j| \neq 1$. Hence for any β constraint (50) describes a generalized Pauli principle:

$$|k_i - k_j| \geqslant |\beta|. \tag{56}$$

When a set of pseudomomentum eigenvalues $\{k_i\}$ is known, (6) gives the energy eigenvalue as

$$E = \sum_{i=1}^{N} k_i^2. \tag{57}$$

This equation shows that the set of $k_i = \alpha_i$ $(i = 1, 2, \ldots, N)$ gives the lowest energy and (48) is the true ground state. In an arbitrary set $\{k_i\}$, the energy E is invariant under an exchange of k_i's. The exchanged set gives a state with the same energy as the original if $|k_i - k_j| \neq |\beta|$ $(i \neq j)$. Thus the braid-exclusion operators $\{X_{i,i+1}\}$ create degenerate states by repeating (54). The ground state is not degenerate, since the operation of $\{X_{i,i+1}\}$ to the ground state $(k_i = \alpha_i)$ gives 0 due to (54).

The degeneracy of an energy eigenvalue is given by counting the number of possible combinations of the corresponding set $\{k_i\}$. We take out all the quantum numbers $m_1, m_2, \ldots, m_L$ which are included in $\{n_j\}$ and are different from each other. Then we define l_i for each m_i so that l_i is the number of elements equal to m_i in $\{n_j\}$. In terms of $\{l_j\}$ the degeneracy is given by

$$\frac{N!}{l_1! l_2! \ldots l_L!}. \tag{58}$$

Thus we have reproduced all the eigenenergies and their degeneracy for the Sutherland model.

In summary, we have found a novel algebraic formalism for the eigenvalue problem of the Sutherland model. All the energy eigenstates are obtained as eigenstates of pseudomomenta $\{D_i\}$. The formalism is based on raising operators $\{b_i^\dagger\}$ and braid-exclusion operators $\{X_{i,i+1}\}$ as well as pseudomomenta $\{D_i\}$. While $b_i^\dagger$ creates another state with different energy, $X_{i,i+1}$ creates another degenerate state. The calculation of the correlation function in the present formalism is a future problem.

We would like to thank Yoshio Ohnuki and Shinsaku Kitakado for useful discussions. This work is partially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture, Japan.

References

[1] Sutherland B 1971 *J. Math. Phys.* **12** 246
Sutherland B 1971 *J. Math. Phys.* **12** 251
Sutherland B 1971 *Phys. Rev.* A **4** 2019
Sutherland B 1972 *Phys. Rev.* A **5** 1372
[2] Jack H 1970 *Proc. R. Soc.* **69** 1
[3] Stanley R 1986 *Adv. Math.* **77** 76

[4] Macdonald I 1979 *Symmetric Function and Hall Polynomials* (New York: Oxford University Press)
[5] Dunkl C 1989 *Trans. Am. Math. Soc.* **311** 167
[6] Cherednik I 1991 *Inv. Math.* **106** 411
[7] Kawakami N 1993 *Phys. Rev. Lett.* **71** 275
Kuramoto Y and Kawakami N *Preprint* (YITP-K-1065)
[8] Laughlin R B 1990 *The Quantum Hall Effect* (Berlin: Springer)
[9] Mimachi K and Yamada Y 1995 *Commun. Math. Phys.* **174** 447
Awata H, Matsuo Y, Odake S and Shiraishi J 1995 *Nucl. Phys.* B **449** 347
[10] Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937
[11] Haldane F D M and Zirnbauer M 1994 *Phys. Rev. Lett.* **71** 4055
Ha Z N C 1994 *Phys. Rev. Lett.* **73** 1574
[12] Simons B and Altshuler B 1993 *Phys. Rev. Lett.* **70** 4063
Simons B, Lee P and Altshuler B 1993 *Phys. Rev. Lett.* **73** 4122
[13] Haldane F D M and Zirnbauer M 1994 *Phys. Rev. Lett.* **71** 4055
[14] Ha Z N 1994 *Phys. Rev. Lett.* **73** 1574
Lesage F, Pasquier V and Serban D 1995 *Nucl. Phys.* B **435** 585
Minahan J and Polychronakos A 1994 *Phys. Rev.* B **50** 4236
[15] Ujino H and Wadati M 1996 *J. Phys. Soc. Japan* **65** 2423
[16] Kakei S 1996 *J. Phys. A: Math. Gen.* **29** L619
Kakei S 1997 *Preprint* q-alg/9706019
[17] Lapointe L and Vinet L 1996 *Commun. Math. Phys.* **17** B 425
[18] Knop F and Sahi S 1997 *Invent. Math.* **128** 9
[19] Kirillov A N and Noumi M 1996 *Preprint* q-alg/9605004
[20] Haldane F D M 1988 *Phys. Rev. Lett.* **60** 635
Shastry B 1988 *Phys. Rev. Lett.* **60** 639
Haldane F D M 1991 *Phys. Rev. Lett.* **66** 1529
Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937
[21] Haldane F D M, Ha Z N, Talstra J, Bernard D and Pasquier V 1992 *Phys. Rev. Lett.* **69** 2021
Bernard D, Gaudin M, Haldane F D M and Pasquier V 1993 *J. Phys. A: Math. Gen.* **26** 5219
[22] Wilczek F 1990 *Fractional Statistics and Anyon Superconductivity* (Singapore: World Scientific)
[23] Ohnuki Y and Kitakado S 1992 *Mod. Phys. Lett.* A **7** 2477
Ohnuki Y and Kitakado S 1993 *J. Math. Phys.* **34** 2827